

Capacity of the Gaussian Two-way Relay Channel to within $\frac{1}{2}$ Bit

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Abstract

In this paper, a Gaussian two-way relay channel, where two source nodes exchange messages with each other through a relay, is considered. We assume that all nodes operate in full-duplex mode and there is no direct channel between the source nodes. We propose an achievable scheme composed of nested lattice codes for the uplink and structured binning for the downlink. We show that the scheme achieves within $\frac{1}{2}$ bit from the cut-set bound for all channel parameters and becomes asymptotically optimal as the signal to noise ratios increase.

Index Terms

Two-way relay channel, wireless networks, network coding, lattice codes

I. INTRODUCTION

We consider a two-way relay channel (TRC), as shown in Fig. 1 (a). Nodes 1 and 2 want to exchange messages with each other, and a relay node facilitates the communication between them. This TRC can be thought of as a basic building block of general wireless networks, along with the relay channel [1], the two-way channel [2], etc. Recently, there have been a great deal of interest in the capacity of wireless networks. Inspired by network coding [3], TRC has been studied in the context of network coding for wireless networks due to its simple structure. However, the capacity region of the general TRC is still unknown.

In [4], several classical relaying strategies for the one-way relay channel [1], such as amplify-and-forward (AF), decode-and-forward (DF), and compress-and-forward (CF), were extended and applied to the TRC. AF relaying is a very simple and practical strategy, but due to the noise amplification, it cannot be optimal in throughput at low signal to noise ratios (SNRs). DF relaying requires the relay to decode all the source messages and, thus, does not suffer from the noise amplification. In [5], it was shown that the achievable rate region of DF relaying can be improved by applying network coding to the decoded messages at the relay. This scheme is sometimes optimal in its throughput [7], but it is generally subject to the *multiplexing loss* [6].

In general, in relay networks, the relay nodes need not reconstruct all the messages, but only need to pass sufficient information to the destination nodes to do so. CF or partial DF relaying strategies for the TRC, in which the relay does not decode the source messages, were studied in [8], [9]. It was shown that these strategies achieve the information theoretic cut-set bound [23] within a constant number of bits when applied to the Gaussian TRC. In [10], [11], structured schemes that use lattice codes were proposed for the Gaussian TRC, and it was shown that these schemes can achieve the cut-set bound within $\frac{1}{2}$ bit.

In this paper, we focus on the Gaussian TRC with full-duplex nodes and no direct communication link between the source nodes. Such a Gaussian TRC is shown in Fig. 1 (b), and it is essentially the same as

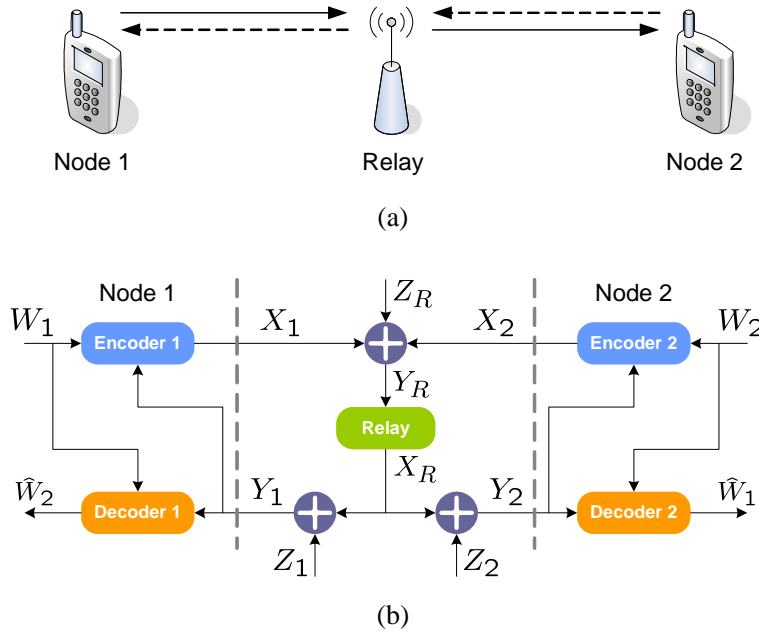


Fig. 1. Gaussian two-way relay channel

those considered in [8]-[11]. For the uplink, i.e., the channel from the source nodes to the relay, we propose a scheme based on nested lattice codes [19] formed from a lattice chain. This scheme is borrowed from the work on the relay networks with interference in [12], [13]. By using nested lattice codes for the uplink, we can exploit the structural gain of *computation coding* [15], which corresponds to a kind of combined channel and network coding. For the downlink, i.e., the channel from the relay to the destination nodes, we see the channel as a BC with receiver side information [7], [16], [17], since the receiver nodes know their own transmitted messages. In such a channel, the capacity region can be achieved by the random binning of messages [16]. In our strategy, a structural binning of messages, rather than the random one, is naturally introduced by the lattice codes used in the uplink. Thus, at each destination node, together with the side information on its own message, this binning can be exploited for decoding.

In fact, as stated above, our work is not the first to apply lattice codes to the Gaussian TRC. However, we assume more a general TRC model compared to the other works. In [11], it was assumed that the channel is symmetric, i.e., all source and relay nodes have the same transmit powers and noise variances. In [10], a lattice scheme for the asymmetric Gaussian TRC was proposed. However, the scheme requires the existence of a certain class of lattices to achieve a $\frac{1}{2}$ bit gap to the cut-set bound. This paper extends those previous works and shows that we can in fact achieve the cut-set bound within $\frac{1}{2}$ bit¹ for any channel parameters, e.g., transmit powers and noise variances. Moreover, the gap vanishes as the uplink SNRs increase.

This paper is organized as follows. In Section II, we present the channel model and define related parameters. The cut-set bound on the capacity region is given in Section III. Section IV illustrates our achievable scheme and computes the achievable rate region. Section V concludes the paper.

II. SYSTEM MODEL

We consider a Gaussian two-way relay channel, as shown in Fig. 1 (b). We assume that the source and relay nodes operate in full-duplex mode and there is no direct path between the two source nodes. The variables of the channel are as follows:

¹To be exact, this implies $\frac{1}{2}$ bit per dimension. For a complex-valued system, as considered in [8], we have 1 bit gap per complex dimension.

- $W_i \in \{1, \dots, 2^{nR_i}\}$: message of node i ,
- $\mathbf{X}_i = [X_i^{(1)}, \dots, X_i^{(n)}]^T$: channel input of node i ,
- $\mathbf{Y}_R = [Y_R^{(1)}, \dots, Y_R^{(n)}]^T$: channel output at the relay,
- $\mathbf{X}_R = [X_R^{(1)}, \dots, X_R^{(n)}]^T$: channel input of the relay,
- $\mathbf{Y}_i = [Y_i^{(1)}, \dots, Y_i^{(n)}]^T$: channel output at node i ,
- $\hat{W}_i \in \{1, \dots, 2^{nR_i}\}$: estimated message of node i ,

where $i \in \{1, 2\}$, n is the number of channel uses, and R_i denotes the rate of node i . We assume that the messages W_1 and W_2 are independent of each other. Node i transmits $X_i^{(t)}$ at time t to the relay through the uplink channel specified by

$$Y_R^{(t)} = X_1^{(t)} + X_2^{(t)} + Z_R^{(t)},$$

where $Z_R^{(t)}$ is an independent identically distributed (i.i.d.) Gaussian random variable with zero mean and variance σ_R^2 . The transmit signal $X_i^{(t)}$ is determined as a function of message W_i and past channel outputs $Y_i^{t-1} = \{Y_i^{(1)}, \dots, Y_i^{(t-1)}\}$, i.e., $X_i^{(t)} = f_i^{(t)}(W_i, Y_i^{t-1})$. There are power constraints P_i , $i \in \{1, 2\}$ on the transmitted signals

$$\frac{1}{n} \sum_{t=1}^n \left(X_i^{(t)}\right)^2 \leq P_i, \quad i = 1, 2.$$

At the same time, the relay transmits $X_R^{(t)}$ to nodes 1 and 2 through the downlink channel specified by

$$Y_i^{(t)} = X_R^{(t)} + Z_i^{(t)}, \quad i \in \{1, 2\},$$

where $Z_i^{(t)}$ is an i.i.d. Gaussian random variable with zero mean and variance σ_i^2 . The power constraint at the relay is given by

$$\frac{1}{n} \sum_{t=1}^n \left(X_R^{(t)}\right)^2 \leq P_R.$$

Since the relay has no messages of its own, $X_R^{(t)}$ is formed as a function of past channel outputs $Y_R^{t-1} = \{Y_R^{(1)}, \dots, Y_R^{(t-1)}\}$, i.e., $X_R^{(t)} = f_R^{(t)}(Y_R^{t-1})$. At node 1, the message estimate $\hat{W}_2 = g_1(W_1, \mathbf{Y}_1)$ is computed from the received signal \mathbf{Y}_1 and its message W_1 . The decoding of node 2 is performed similarly. Now, the average probability of error is defined as

$$P_e = \Pr \left\{ \hat{W}_1 \neq W_1 \text{ or } \hat{W}_2 \neq W_2 \right\}.$$

For the aforementioned TRC, we say that a rate pair (R_1, R_2) is achievable if a sequence of encoding and decoding functions exists such that the error probability vanishes as n tends to infinity. The capacity region of the TRC is defined as the convex closure of all achievable rate pairs.

III. AN UPPER BOUND FOR THE CAPACITY REGION

By the cut-set bound [23], if the rate pair (R_1, R_2) is achievable for a general TRC, a joint probability distribution $p(x_1, x_2, x_R)$ exists such that

$$R_1 \leq \min \{I(X_1; Y_R, Y_2 | X_R, X_2), I(X_1, X_R; Y_2 | X_2)\}, \quad (1a)$$

$$R_2 \leq \min \{I(X_2; Y_R, Y_1 | X_R, X_1), I(X_2, X_R; Y_1 | X_1)\}. \quad (1b)$$

In particular, for the Gaussian TRC, we can use the fact that there is no direct path between nodes 1 and 2, i.e., $(X_1, X_2, Y_R) \rightarrow X_R \rightarrow (Y_1, Y_2)$, and that $X_R \rightarrow (X_1, X_2) \rightarrow Y_R$. This induces another upper bound from (1), given by

$$R_1 \leq \min \{I(X_1; Y_R | X_2), I(X_R; Y_2)\}, \quad (2a)$$

$$R_2 \leq \min \{I(X_2; Y_R | X_1), I(X_R; Y_1)\}, \quad (2b)$$

for some $p(x_1, x_2, x_R)$. It can be easily seen that, for the Gaussian TRC with transmit power constraints, all terms under the minimizations in (2) are maximized by the product distribution $p(x_1, x_2, x_R) = p(x_1)p(x_2)p(x_R)$, where $p(x_1)$, $p(x_2)$, and $p(x_R)$ are Gaussian probability density functions with zero means and variances P_1 , P_2 , and P_R , respectively. The resulting upper bound on the capacity region is given by

$$R_1 \leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{P_1}{\sigma_R^2} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_2^2} \right) \right\}, \quad (3a)$$

$$R_2 \leq \min \left\{ \frac{1}{2} \log \left(1 + \frac{P_2}{\sigma_R^2} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_1^2} \right) \right\}. \quad (3b)$$

IV. AN ACHIEVABLE RATE REGION FOR THE GAUSSIAN TRC

In this section, we compute an achievable rate region for the Gaussian TRC. For the uplink, we consider using nested lattice codes, which are formed from a lattice chain. For the downlink, we use a structured binning of messages at the relay, which is naturally introduced by the nested lattice codes. The destination nodes decode each other's message using this binning and the side information on their own transmitted messages.

The main result of this section is as follows:

Theorem 1: For a Gaussian TRC, as shown in Fig. 1 (b), we can achieve the following region:

$$R_1 \leq \min \left\{ \left[\frac{1}{2} \log \left(\frac{P_1}{P_1 + P_2} + \frac{P_1}{\sigma_R^2} \right) \right]^+, \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_2^2} \right) \right\}, \quad (4a)$$

$$R_2 \leq \min \left\{ \left[\frac{1}{2} \log \left(\frac{P_2}{P_1 + P_2} + \frac{P_2}{\sigma_R^2} \right) \right]^+, \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_1^2} \right) \right\}, \quad (4b)$$

where $[x]^+ \triangleq \max\{x, 0\}$.

Note that the achievable rate region in Theorem 1 is within $\frac{1}{2}$ bit of the upper bound (3), regardless of channel parameters such as the transmit powers and noise variances. Moreover, as the uplink SNRs $\frac{P_1}{\sigma_R^2}$ and $\frac{P_2}{\sigma_R^2}$ increase, the gap vanishes and our achievable region asymptotically approaches the capacity region of the Gaussian TRC.

We prove Theorem 1 in the following subsections.

A. Lattice scheme for the uplink

For the scheme for the uplink, we consider a lattice coding scheme. We will not cover the full details of lattices and lattice codes due to page limitations. For a comprehensive review, we refer the reader to [19]–[21] and the references therein.

A *nested lattice code* is defined in terms of two n -dimensional lattices Λ_C^n and Λ^n , which form a lattice partition Λ_C^n / Λ^n , i.e., $\Lambda^n \subseteq \Lambda_C^n$. The nested lattice code is a lattice code which uses Λ_C^n as codewords and the Voronoi region of Λ^n as a shaping region. For Λ_C^n / Λ^n , we define the *set of coset leaders* as

$$\mathcal{C} = \{\Lambda_C^n \bmod \Lambda^n\} \triangleq \{\Lambda_C^n \cap \mathcal{R}\},$$

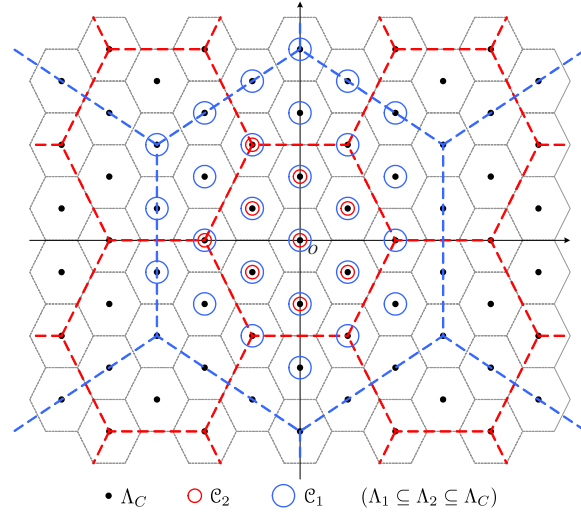


Fig. 2. Example of a lattice chain and sets of coset leaders. $\mathcal{C}_2 \subseteq \mathcal{C}_1 \subseteq \Lambda_C$.

where \mathcal{R} is the Voronoi region of Λ . Then the coding rate of the nested lattice code is given by

$$R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log \frac{\text{Vol}(\Lambda^n)}{\text{Vol}(\Lambda_C^n)},$$

where $\text{Vol}(\cdot)$ denotes the volume of the Voronoi region of a lattice. For the TRC, we should design two nested lattice codes, one for each source node. This subsection will show how the nested lattice codes are formed. In the following argument, we assume that $P_1 \geq P_2$ without loss of generality. Now, let us first consider a theorem that is a key for our code construction.

Theorem 2: For any $P_1 \geq P_2 \geq 0$ and $\gamma \geq 0$, a sequence of n -dimensional lattice chains $\Lambda_1^n \subseteq \Lambda_2^n \subseteq \Lambda_C^n$ exists that satisfies the following properties.

- a) Λ_1^n and Λ_2^n are simultaneously Rogers-good and Poltyrev-good while Λ_C^n is Poltyrev-good (for the notion of goodness of lattices, see [20]).
- b) For any $\epsilon > 0$, $P_i - \epsilon \leq \sigma^2(\Lambda_i^n) \leq P_i$, $i \in \{1, 2\}$, for sufficiently large n , where $\sigma^2(\cdot)$ denotes the second moment per dimension associated with the Voronoi region of the lattice.
- c) The coding rate of the nested lattice code associated with the lattice partition Λ_C^n/Λ_2^n can approach any value as n tends to infinity, i.e.,

$$R_2 = \frac{1}{n} \log |\mathcal{C}_2| = \frac{1}{n} \log \left(\frac{\text{Vol}(\Lambda_2^n)}{\text{Vol}(\Lambda_C^n)} \right) = \gamma + o_n(1), \quad (5)$$

where $\mathcal{C}_2 = \{\Lambda_C^n \bmod \Lambda_2^n\}$ and $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, the coding rate of the nested lattice code associated with Λ_C^n/Λ_1^n is given by

$$R_1 = \frac{1}{n} \log |\mathcal{C}_1| = \frac{1}{n} \log \left(\frac{\text{Vol}(\Lambda_1^n)}{\text{Vol}(\Lambda_C^n)} \right) = R_2 + \frac{1}{2} \log \left(\frac{P_1}{P_2} \right) + o_n(1),$$

where $\mathcal{C}_1 = \{\Lambda_C^n \bmod \Lambda_1^n\}$.

Proof: See the proof of Theorem 2 in [13]. ■

For instance, a lattice chain and the corresponding sets of coset leaders are visualized in Fig. 2 for the two-dimensional case.

Encoding

Let us think of a lattice chain (more precisely, a sequence of lattice chains) and sets of coset leaders as described in Theorem 2. We use \mathcal{C}_1 and \mathcal{C}_2 for nodes 1 and 2 respectively. For node i , the message

set $\{1, \dots, 2^{nR_i}\}$ is one-to-one mapped to \mathcal{C}_i . Thus, to transmit a message, node i chooses $\mathbf{W}_i \in \mathcal{C}_i$ associated with the message and sends

$$\mathbf{X}_i = (\mathbf{W}_i + \mathbf{U}_i) \bmod \Lambda_i,$$

where \mathbf{U}_i is a random dither vector with $\mathbf{U}_i \sim \text{Unif}(\mathcal{R}_i)$ and \mathcal{R}_i denotes the Voronoi region of Λ_i (we suppressed the superscript “ n ” for simplicity). The dither vectors \mathbf{U}_i , $i \in \{1, 2\}$, are independent of each other and also independent of the messages and the noise. We assume that each \mathbf{U}_i is known to the source nodes and the relay. Note that, due to the *crypto-lemma* [21], \mathbf{X}_i is uniformly distributed over \mathcal{R}_i and independent of \mathbf{W}_i . Thus, the average transmit power of node i is equal to $\sigma^2(\Lambda_i)$, which approaches P_i as n tends to infinity, and the power constraint is met.

Decoding

The received vector at the relay is given by

$$\mathbf{Y}_R = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{Z}_R,$$

where $\mathbf{Z}_R = [Z_R^{(1)}, \dots, Z_R^{(n)}]^T$. Upon receiving \mathbf{Y}_R , the relay computes

$$\begin{aligned} \tilde{\mathbf{Y}}_R &= \left(\alpha \mathbf{Y}_R - \sum_{j=1}^2 \mathbf{U}_j \right) \bmod \Lambda_1 \\ &= \left[\sum_{j=1}^2 (\mathbf{W}_j + \mathbf{U}_j) \bmod \Lambda_j - \sum_{j=1}^2 \mathbf{X}_j \right. \\ &\quad \left. + \alpha \sum_{j=1}^2 \mathbf{X}_j + \alpha \mathbf{Z}_R - \sum_{j=1}^2 \mathbf{U}_j \right] \bmod \Lambda_1 \\ &= (\mathbf{T} + \tilde{\mathbf{Z}}_R) \bmod \Lambda_1, \end{aligned}$$

where

$$\begin{aligned} \mathbf{T} &= \left[\sum_{j=1}^2 (\mathbf{W}_j - Q_j(\mathbf{W}_j + \mathbf{U}_j)) \right] \bmod \Lambda_1 \\ &= [\mathbf{W}_1 + \mathbf{W}_2 - Q_2(\mathbf{W}_2 + \mathbf{U}_2)] \bmod \Lambda_1, \end{aligned} \tag{6}$$

$$\tilde{\mathbf{Z}}_R = -(1 - \alpha)(\mathbf{X}_1 + \mathbf{X}_2) + \alpha \mathbf{Z}_R, \tag{7}$$

$\alpha \in [0, 1]$ is a scaling factor, and $Q_j(\cdot)$ denotes the nearest neighbor lattice quantizer associated with Λ_j . If we let α be the minimum mean-square error (MMSE) coefficient

$$\alpha = \frac{P_1 + P_2}{P_1 + P_2 + \sigma_R^2},$$

the variance of the effective noise (7) satisfies

$$\frac{1}{n} E \left\{ \left\| \tilde{\mathbf{Z}}_R \right\|^2 \right\} \leq \frac{(P_1 + P_2) \sigma_R^2}{P_1 + P_2 + \sigma_R^2}.$$

From the chain relation of lattices in Theorem 2, it follows that $\mathbf{T} \in \mathcal{C}_1$. Moreover, using the crypto-lemma, it is obvious that \mathbf{T} is uniformly distributed over \mathcal{C}_1 and independent of $\tilde{\mathbf{Z}}_R$ [13, Lemma 2].

The relay attempts to recover \mathbf{T} from $\tilde{\mathbf{Y}}_R$ instead of recovering \mathbf{W}_1 and \mathbf{W}_2 separately. Thus, the lattice scheme inherits the idea of computation coding [15] and physical-layer network coding [18]. Also, by not requiring the relay to decode both messages, \mathbf{W}_1 and \mathbf{W}_2 , we can avoid the multiplexing loss [5] at the relay. The method of decoding we consider is *Euclidean lattice decoding* [19]–[22], which

finds the closest point to $\tilde{\mathbf{Y}}_R$ in Λ_C . Thus, the estimate of \mathbf{T} is given by $\hat{\mathbf{T}} = Q_C(\tilde{\mathbf{Y}})$, where $Q_C(\cdot)$ denotes the nearest neighbor lattice quantizer associated with Λ_C . Then, from the lattice symmetry and the independence between \mathbf{T} and $\tilde{\mathbf{Z}}_R$, the probability of decoding error is given by

$$\begin{aligned} p_e &= \Pr \left\{ \hat{\mathbf{T}} \neq \mathbf{T} \right\} \\ &= \Pr \left\{ \tilde{\mathbf{Z}}_R \bmod \Lambda_1 \notin \mathcal{R}_C \right\}, \end{aligned} \quad (8)$$

where \mathcal{R}_C denotes the Voronoi region of Λ_C . We then have the following theorem.

Theorem 3: Let

$$R_1^* = \left\lceil \frac{1}{2} \log \left(\frac{P_1}{P_1 + P_2} + \frac{P_1}{\sigma_R^2} \right) \right\rceil^+.$$

For any $\bar{R}_1 < R_1^*$ and a lattice chain as described in Theorem 2 with R_1 approaching \bar{R}_1 , i.e., $R_1 = \bar{R}_1 + o_n(1)$, the error probability under Euclidean lattice decoding (8) is bounded by

$$p_e \leq e^{-n(E_P(2^{2(R_1^* - \bar{R}_1)} - o_n(1)))},$$

where $E_P(\cdot)$ is the Poltyrev exponent [22].

Proof: See the proof of Theorem 3 in [13]. ■

According to Theorem 3, the error probability vanishes as $n \rightarrow \infty$ if $\bar{R}_1 < R_1^*$ since $E_P(x) > 0$ for $x > 1$. This implies that the nested lattice code can have any rate below R_1^* for the reliable decoding of \mathbf{T} . Thus, by c) of Theorem 2 and Theorem 3, the error probability at the relay vanishes as $n \rightarrow \infty$ if

$$R_i < \left\lceil \frac{1}{2} \log \left(\frac{P_i}{P_1 + P_2} + \frac{P_i}{\sigma_R^2} \right) \right\rceil^+, \quad i = 1, 2. \quad (9)$$

B. Downlink phase

We first generate 2^{nR_1} n -sequences with each element i.i.d. according to $\mathcal{N}(0, P_R)$. These sequences form a codebook \mathcal{C}_R . We assume one-to-one correspondence between each $\mathbf{t} \in \mathcal{C}_1$ and a codeword $\mathbf{X}_R \in \mathcal{C}_R$. To make this correspondence explicit, we use the notation $\mathbf{X}_R(\mathbf{t})$. After the relay decodes $\hat{\mathbf{T}}$, it transmits $\mathbf{X}_R(\hat{\mathbf{T}})$ at the next block to nodes 1 and 2. We now assume that there is no error in the uplink, i.e., $\hat{\mathbf{T}} = \mathbf{T}$. Under this condition, $\hat{\mathbf{T}}$ is uniform over \mathcal{C}_1 , and, thus, $\mathbf{X}_R(\hat{\mathbf{T}})$ is also uniformly chosen from \mathcal{C}_R .

Upon receiving $\mathbf{Y}_1 = \mathbf{X}_R + \mathbf{Z}_1$, where $\mathbf{Z}_1 = [Z_1^{(1)}, \dots, Z_1^{(n)}]^T$, node 1 estimates the relay message $\hat{\mathbf{T}}$ as $\hat{\mathbf{T}}_1 = \mathbf{t}_1$ if a unique codeword exists in $\mathcal{C}_{R,1}$ such that $(\mathbf{X}_R(\mathbf{t}_1), \mathbf{Y}_1)$ are jointly typical, where

$$\mathcal{C}_{R,1} = \{ \mathbf{X}_R(\mathbf{t}) : \mathbf{t} = [\mathbf{W}_1 + \mathbf{w}_2 - Q_2(\mathbf{w}_2 + \mathbf{U}_2)] \bmod \Lambda_1, \mathbf{w}_2 \in \mathcal{C}_2 \}.$$

Then, from the knowledge of \mathbf{W}_1 and $\hat{\mathbf{T}}_1$, node 1 estimates the message of node 2 as

$$\hat{\mathbf{W}}_2 = (\hat{\mathbf{T}}_1 - \mathbf{W}_1) \bmod \Lambda_2. \quad (10)$$

Given $\hat{\mathbf{T}} = \mathbf{T}$, we have $\hat{\mathbf{W}}_2 = \mathbf{W}_2$ if and only if $\hat{\mathbf{T}}_1 = \hat{\mathbf{T}}$. Note that $|\mathcal{C}_{R,1}| = 2^{nR_2}$. Thus, from the argument of random coding and jointly typical decoding [23], we have

$$\Pr \left\{ \hat{\mathbf{T}}_1 \neq \hat{\mathbf{T}} | \hat{\mathbf{T}} = \mathbf{T} \right\} \rightarrow 0 \quad (11)$$

as $n \rightarrow \infty$ if

$$R_2 < \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_1^2} \right). \quad (12)$$

Similarly, at node 2, the relay message is estimated to be $\hat{\mathbf{T}}_2 = \mathbf{t}_2$ by finding a unique codeword in $\mathcal{C}_{R,2}$ such that $(\mathbf{X}_R(\mathbf{t}_2), \mathbf{Y}_2)$ are jointly typical, where

$$\mathcal{C}_{R,2} = \{\mathbf{X}_R(\mathbf{t}) : \mathbf{t} = [\mathbf{w}_1 + \mathbf{W}_2 - Q_2(\mathbf{W}_2 + \mathbf{U}_2)] \bmod \Lambda_1, \mathbf{w}_1 \in \mathcal{C}_1\}.$$

Then the message of node 1 is estimated as

$$\hat{\mathbf{W}}_1 = \left[\hat{\mathbf{T}}_2 - \mathbf{W}_2 + Q_2(\mathbf{W}_2 + \mathbf{U}_2) \right] \bmod \Lambda_1. \quad (13)$$

Since $|\mathcal{C}_{R,1}| = 2^{nR_1}$, we have

$$\Pr \left\{ \hat{\mathbf{T}}_2 \neq \hat{\mathbf{T}} | \hat{\mathbf{T}} = \mathbf{T} \right\} \rightarrow 0 \quad (14)$$

as $n \rightarrow \infty$ if

$$R_1 < \frac{1}{2} \log \left(1 + \frac{P_R}{\sigma_2^2} \right). \quad (15)$$

Note that, in the downlink, although the channel setting is broadcast, nodes 1 and 2 achieve their point-to-point channel capacities (12) and (15) without being affected by each other. This is because of the side information on the transmitted message at each node and the binning of message. In our scheme, the relation in (6) represents how the message pair $(\mathbf{W}_1, \mathbf{W}_2)$ is binned to \mathbf{T} .

C. Achievable rate region

Clearly, the message estimates (10) and (13) are exact if and only if $\hat{\mathbf{T}}_1 = \hat{\mathbf{T}}_2 = \mathbf{T}$. Thus, the error probability is given by

$$\begin{aligned} P_e &= \Pr \left\{ \hat{\mathbf{T}}_1 \neq \mathbf{T} \text{ or } \hat{\mathbf{T}}_2 \neq \mathbf{T} \right\} \\ &\leq \Pr \left\{ \hat{\mathbf{T}}_1 \neq \hat{\mathbf{T}} \text{ or } \hat{\mathbf{T}}_2 \neq \hat{\mathbf{T}} \text{ or } \hat{\mathbf{T}} \neq \mathbf{T} \right\} \\ &\leq \Pr \left\{ \hat{\mathbf{T}} \neq \mathbf{T} \right\} + \Pr \left\{ \hat{\mathbf{T}}_1 \neq \hat{\mathbf{T}} | \hat{\mathbf{T}} = \mathbf{T} \right\} + \Pr \left\{ \hat{\mathbf{T}}_2 \neq \hat{\mathbf{T}} | \hat{\mathbf{T}} = \mathbf{T} \right\} \end{aligned} \quad (16)$$

By Theorem 3, the first term of (16) vanishes as $n \rightarrow \infty$ if $R_i < R_i^*$, $i \in \{1, 2\}$. Also, by (11) and (14), the second and third terms also vanish as $n \rightarrow \infty$ if (12) and (15) hold. Thus, the achievable rate region (4) follows from (9), (12), and (15).

V. CONCLUSION

In this paper, we considered the Gaussian TRC. An achievable scheme was presented based on nested lattice codes for the uplink and structured binning for the downlink. The resulting achievable rate region approaches to within $\frac{1}{2}$ bit of the cut-set bound for all channel parameters, and the gap eventually vanishes in the high SNR regime. Though the capacity region is very nearly reached, the exact capacity region of the Gaussian TRC is still an open problem.

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